

# Fermi-Bose Correspondence at Finite Temperature

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The correspondence between fermi-sea/bose-condensate displacements and the number-conserving product of two fermi/bose fields is generalised to finite temperatures. It is shown that the straightforward generalisation that involves making the sea-bosons participate in the thermodynamic averaging does not work out. A remedy is found which involves treating the sea-displacement bosons at zero temperature while all finite temperature effects have to be lumped into the coefficients. We also show that all the finite temperature dynamical four-point and six-point functions come out correctly as do some of the commutation rules involving the fermion/boson product that go through unscathed. It is also shown that this unusual prescription of leaving out the sea-bosons from the thermodynamic averaging does in fact reproduce the correct RPA dielectric function at finite temperature. This article is not however, critic-proof.

## I. INTRODUCTION

The attempts of recent preprints have been to construct a nonperturbative theory of interacting fermions/bosons. In order for this to have practical consequences, we would like to be able to generalise these ideas to finite temperatures where one might be able to study important physical phenomena such as phase transitions. The generalisation to finite temperatures is not as straightforward as it seems. The obvious generalisation that involves making the sea-displacement bosons participate in the thermodynamic averaging while retaining the original formula that relates the fermion/boson product to the sea-displacements does not work out. This is explained in detail and a remedy is found. The remedy involves making the following unusual prescription. The sea-displacement bosons are assumed to remain at zero temperature and do not participate in the thermodynamic averaging. Rather, the finite temperature effects have to be lumped into the coefficients thereby altering the relation that connects the fermion/boson product and sea-displacements. It must be made sure that in the bargain, commutation rules involving the number-conserving products are not altered. All this is shown explicitly in the next few sections.

## II. FERMION PRODUCT AND SEA-DISPLACEMENT CORRESPONDENCE

To motivate the development of the finite temperature correspondence let us first try to argue that the naive generalisation that seems very natural does not work out. One would expect from our earlier preprint<sup>1</sup>

that at finite temperature, the correlation functions in the bose language are merely thermodynamic expectation values of the corresponding dynamical quantities. This does not seem to work out as this simple demonstration shows. Take for example, the momentum distribution at finite temperature,

$$\begin{aligned} \langle c_{\mathbf{k}}^{\dagger} c_{\mathbf{k}} \rangle &= n_F(\mathbf{k}) + \sum_{\mathbf{q}_1} \Lambda_{\mathbf{k}-\mathbf{q}_1/2}(-\mathbf{q}_1) \langle a_{\mathbf{k}-\mathbf{q}_1/2}^{\dagger}(\mathbf{q}_1) a_{\mathbf{k}-\mathbf{q}_1/2}(\mathbf{q}_1) \rangle \\ &\quad - \sum_{\mathbf{q}_1} \Lambda_{\mathbf{k}+\mathbf{q}_1/2}(-\mathbf{q}_1) \langle a_{\mathbf{k}+\mathbf{q}_1/2}^{\dagger}(\mathbf{q}_1) a_{\mathbf{k}+\mathbf{q}_1/2}(\mathbf{q}_1) \rangle \end{aligned} \quad (1)$$

Since,

$$\langle c_{\mathbf{k}}^{\dagger} c_{\mathbf{k}} \rangle = \frac{\text{tr}(e^{-\beta(H-\mu_F N)} c_{\mathbf{k}}^{\dagger} c_{\mathbf{k}})}{\text{tr}(e^{-\beta(H-\mu_F N)})} \quad (2)$$

Assume that the  $n_F(\mathbf{k})$  are all evaluated at zero temperature as in the previous preprints. Taken at face value, one is obliged to compute the thermodynamic expectation values of the bose occupation probabilities assuming the chemical potential for the bosons is zero as the total number of fermions commutes with the sea-displacements (this means that we are allowed to create and destroy any number of bosons). Now such a calculation yields an infinite answer for  $\langle c_{\mathbf{k}}^{\dagger} c_{\mathbf{k}} \rangle$  as the sum over all  $\mathbf{q}_1$  diverges (is proportional to the volume of the system). This means that we are no longer allowed to treat the sea-displacements bosons as participating in the thermodynamic averaging. Rather, they merely provide the correct time-evolution of the product of fermi fields, the thermodynamic aspect has to be absorbed into the coefficients  $n_F(\mathbf{k})$  and  $\Lambda_{\mathbf{k}}(\mathbf{q})$ , as we shall soon see. For this, let us repostulate a form for the product of fermi fields and argue why this should be as it is. The only guide is that all the finite temperature dynamical correlation functions involving the number conserving object  $c_{\mathbf{k}+\mathbf{q}/2}^{\dagger} c_{\mathbf{k}-\mathbf{q}/2}$  should be correctly reproduced. The commutation rules involving these number conserving products should not be damaged in the bargain. Let us start with the simplest case namely,  $\langle c_{\mathbf{k}+\mathbf{q}/2}^{\dagger} c_{\mathbf{k}-\mathbf{q}/2} \rangle$ . We know what the answer should be, that is,

$$\langle c_{\mathbf{k}+\mathbf{q}/2}^{\dagger} c_{\mathbf{k}-\mathbf{q}/2} \rangle = \delta_{\mathbf{q}=0} n_{F,\beta}(\mathbf{k}) \quad (3)$$

where,

$$n_{F,\beta}(\mathbf{k}) = [\exp(\beta(\epsilon_{\mathbf{k}} - \mu_F)) + 1]^{-1} \quad (4)$$

For this to happen we have to do the following,

- (1) Treat the bosons as before assuming that they are always at zero temperature and with zero chemical potential.
- (2) Assume that all finite temperature effects are lumped into the coefficients.
- (3) Fix the coefficients as before by demanding that the finite temperature dynamical moments of the number-conserving products of fermi fields come out right. In order to cut a long story short, let us merely postulate the form of the product and later on argue why this should be as it is. The following formulas are

true ONLY for temperatures above zero. At temperatures exactly equal to zero the answers have already been given elsewhere<sup>1</sup>. For  $\mathbf{q} \neq 0$  we have,

$$\begin{aligned}
c_{\mathbf{k}+\mathbf{q}/2}^\dagger c_{\mathbf{k}-\mathbf{q}/2} &= \left(\sqrt{\frac{N}{\langle N \rangle}}\right) [\Lambda_{\mathbf{k}}^\beta(\mathbf{q}) a_{\mathbf{k}}(-\mathbf{q}) + \Lambda_{\mathbf{k}}^\beta(-\mathbf{q}) a_{\mathbf{k}}^\dagger(\mathbf{q})] \\
&+ T_1(\mathbf{k}, \mathbf{q}) \sum_{\mathbf{q}_1 \neq 0} a_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2}^\dagger(\mathbf{q}_1) a_{\mathbf{k}-\mathbf{q}_1/2}(\mathbf{q}_1 - \mathbf{q}) \\
&- T_2(\mathbf{k}, \mathbf{q}) \sum_{\mathbf{q}_1 \neq 0} a_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2}^\dagger(\mathbf{q}_1) a_{\mathbf{k}+\mathbf{q}_1/2}(\mathbf{q}_1 - \mathbf{q})
\end{aligned} \tag{5}$$

Here,

$$\begin{aligned}
T_1(\mathbf{k}, \mathbf{q}) &= \sqrt{(1 - n_{F,\beta}(\mathbf{k} + \mathbf{q}/2))} \sqrt{(1 - n_{F,\beta}(\mathbf{k} - \mathbf{q}/2))} \\
T_2(\mathbf{k}, \mathbf{q}) &= \sqrt{n_{F,\beta}(\mathbf{k} + \mathbf{q}/2) n_{F,\beta}(\mathbf{k} - \mathbf{q}/2)} \\
\Lambda_{\mathbf{k}}^\beta(\mathbf{q}) &= \sqrt{n_{F,\beta}(\mathbf{k} + \mathbf{q}/2) (1 - n_{F,\beta}(\mathbf{k} - \mathbf{q}/2))}
\end{aligned} \tag{6}$$

It can be shown quite easily that all the dynamical four and six-point functions of the object  $c_{\mathbf{k}+\mathbf{q}/2}^\dagger c_{\mathbf{k}-\mathbf{q}/2}$  are recovered exactly (See Appendix). For  $\mathbf{q} = 0$  we have to find another expression in order for the kinetic energy operator to come out right. For this let us postulate a form and then compute the coefficients.

$$\begin{aligned}
c_{\mathbf{k}}^\dagger c_{\mathbf{k}} &= n_{F,\beta}(\mathbf{k}) + \sum_{\mathbf{q}_1 \neq 0} S_1(\mathbf{k}, \mathbf{q}_1) a_{\mathbf{k}-\mathbf{q}_1/2}^\dagger(\mathbf{q}_1) a_{\mathbf{k}-\mathbf{q}_1/2}(\mathbf{q}_1) \\
&- \sum_{\mathbf{q}_1 \neq 0} S_2(\mathbf{k}, \mathbf{q}_1) a_{\mathbf{k}+\mathbf{q}_1/2}^\dagger(\mathbf{q}_1) a_{\mathbf{k}+\mathbf{q}_1/2}(\mathbf{q}_1)
\end{aligned} \tag{7}$$

The coefficients  $S_1$  and  $S_2$  here have to be chosen so as to recover the following form of the kinetic energy operator. The latter is necessary to ensure that the time-evolution of the various operators come out right. The correct form of the kinetic energy operator is therefore (taking a cue from the zero temperature case<sup>1</sup>),

$$K = \sum_{\mathbf{k}, \mathbf{q}} \left(\frac{\mathbf{k} \cdot \mathbf{q}}{m}\right) a_{\mathbf{k}}^\dagger(\mathbf{q}) a_{\mathbf{k}}(\mathbf{q}) \tag{8}$$

This means,

$$\epsilon_{\mathbf{k}+\mathbf{q}/2} S_1(\mathbf{k} + \mathbf{q}/2, \mathbf{q}) - \epsilon_{\mathbf{k}-\mathbf{q}/2} S_2(\mathbf{k} - \mathbf{q}/2, \mathbf{q}) = \left(\frac{\mathbf{k} \cdot \mathbf{q}}{m}\right) \tag{9}$$

The obvious solution to this is,

$$S_1(\mathbf{k} + \mathbf{q}/2, \mathbf{q}) = S_2(\mathbf{k} - \mathbf{q}/2, \mathbf{q}) = 1 \tag{10}$$

Thus we may rewrite the above formula for the occupation number operator as,

$$c_{\mathbf{k}}^\dagger c_{\mathbf{k}} = n_{F,\beta}(\mathbf{k}) + \sum_{\mathbf{q}_1 \neq 0} a_{\mathbf{k}-\mathbf{q}_1/2}^\dagger(\mathbf{q}_1) a_{\mathbf{k}-\mathbf{q}_1/2}(\mathbf{q}_1) - \sum_{\mathbf{q}_1 \neq 0} a_{\mathbf{k}+\mathbf{q}_1/2}^\dagger(\mathbf{q}_1) a_{\mathbf{k}+\mathbf{q}_1/2}(\mathbf{q}_1) \quad (11)$$

It is straightforward to show that these formulas reproduce the correct commutation rules described below ( $\mathbf{q} \neq 0$  and  $\mathbf{q}' \neq 0$ ) at least in so far as the terms linear in the sea-displacements are concerned. The claim is that practical results such as the answer for the dynamical current-current and density-density correlations should have the right form even though many of the nuances may not be right. Think about it, we have a whole class of exactly solvable models in any number of spatial dimensions and are able to recover RPA and all the other well-known results in terms of quadratures, a feature that is likely to persist even after introducing more complicated physical processes such as interaction with phonons.

$$[c_{\mathbf{k}'}^\dagger c_{\mathbf{k}'}, c_{\mathbf{k}+\mathbf{q}/2}^\dagger c_{\mathbf{k}-\mathbf{q}/2}] = c_{\mathbf{k}'}^\dagger c_{\mathbf{k}-\mathbf{q}/2} \delta_{\mathbf{k}', \mathbf{k}+\mathbf{q}/2} - c_{\mathbf{k}+\mathbf{q}/2}^\dagger c_{\mathbf{k}'} \delta_{\mathbf{k}', \mathbf{k}-\mathbf{q}/2} \quad (12)$$

and,

$$[c_{\mathbf{k}+\mathbf{q}/2}^\dagger c_{\mathbf{k}-\mathbf{q}/2}, c_{\mathbf{k}'+\mathbf{q}'/2}^\dagger c_{\mathbf{k}'-\mathbf{q}'/2}] = c_{\mathbf{k}+\mathbf{q}/2}^\dagger c_{\mathbf{k}'-\mathbf{q}'/2} \delta_{\mathbf{k}-\mathbf{q}/2, \mathbf{k}'+\mathbf{q}'/2} - c_{\mathbf{k}'+\mathbf{q}'/2}^\dagger c_{\mathbf{k}-\mathbf{q}/2} \delta_{\mathbf{k}'-\mathbf{q}'/2, \mathbf{k}+\mathbf{q}/2} \quad (13)$$

### A. Finite Temperature RPA

In this subsection, we demonstrate that the RPA dielectric function is recovered exactly by selectively retaining parts of the coulomb interaction that lead to RPA. We know that the kinetic energy in the bose language is given by,

$$H_{kin} = \sum_{\mathbf{k}, \mathbf{q}} \left( \frac{\mathbf{k} \cdot \mathbf{q}}{m} \right) a_{\mathbf{k}}^\dagger(\mathbf{q}) a_{\mathbf{k}}(\mathbf{q}) \quad (14)$$

For this let us choose,

$$H_I = \sum_{\mathbf{q} \neq 0} \frac{v_{\mathbf{q}}}{2V} \tilde{\rho}_{\mathbf{q}} \tilde{\rho}_{-\mathbf{q}} \quad (15)$$

where,

$$\tilde{\rho}_{\mathbf{q}} = \sum_{\mathbf{k}} [\Lambda_{\mathbf{k}}^\beta(\mathbf{q}) a_{\mathbf{k}}(-\mathbf{q}) + \Lambda_{\mathbf{k}}^\beta(-\mathbf{q}) a_{\mathbf{k}}^\dagger(\mathbf{q})] \quad (16)$$

From this it may be shown that the RPA dielectric function is recovered as the following demonstration shows. Assume that a weak time-varying external perturbation is applied as shown below,

$$H_{ext} = \sum_{\mathbf{q} \neq 0} (U_{ext}(\mathbf{q}, t) + U_{ext}^*(-\mathbf{q}, t)) \tilde{\rho}_{\mathbf{q}} \quad (17)$$

here,

$$U_{ext}(\vec{r}, t) = U_0 e^{i\mathbf{q} \cdot \vec{r} - i\omega t} \quad (18)$$

Let us now write down the equations of motion for the various boson fields,

$$\begin{aligned} i \frac{\partial}{\partial t} \langle a_{\mathbf{k}}^t(\mathbf{q}) \rangle &= \omega_{\mathbf{k}}(\mathbf{q}) \langle a_{\mathbf{k}}^t(\mathbf{q}) \rangle + \left( \frac{v_{\mathbf{q}}}{V} \right) \Lambda_{\mathbf{k}}^{\beta}(-\mathbf{q}) \sum_{\mathbf{k}'} [\Lambda_{\mathbf{k}'}^{\beta}(-\mathbf{q}) \langle a_{\mathbf{k}'}^t(\mathbf{q}) \rangle + \Lambda_{\mathbf{k}'}^{\beta}(\mathbf{q}) \langle a_{\mathbf{k}'}^{t\dagger}(-\mathbf{q}) \rangle] \\ &+ (U_{ext}(\mathbf{q}, t) + U_{ext}^*(-\mathbf{q}, t)) \Lambda_{\mathbf{k}}^{\beta}(-\mathbf{q}) \end{aligned} \quad (19)$$

$$\begin{aligned} -i \frac{\partial}{\partial t} \langle a_{\mathbf{k}}^{t\dagger}(-\mathbf{q}) \rangle &= \omega_{\mathbf{k}}(-\mathbf{q}) \langle a_{\mathbf{k}}^{t\dagger}(-\mathbf{q}) \rangle + \left( \frac{v_{\mathbf{q}}}{V} \right) \Lambda_{\mathbf{k}}^{\beta}(\mathbf{q}) \sum_{\mathbf{k}'} [\Lambda_{\mathbf{k}'}^{\beta}(-\mathbf{q}) \langle a_{\mathbf{k}'}^t(\mathbf{q}) \rangle + \Lambda_{\mathbf{k}'}^{\beta}(\mathbf{q}) \langle a_{\mathbf{k}'}^{t\dagger}(-\mathbf{q}) \rangle] \\ &+ (U_{ext}(\mathbf{q}, t) + U_{ext}^*(-\mathbf{q}, t)) \Lambda_{\mathbf{k}}^{\beta}(\mathbf{q}) \end{aligned} \quad (20)$$

Now let us decompose the expectation values as follows,

$$\langle a_{\mathbf{k}}^t(\mathbf{q}) \rangle = U_{ext}(\mathbf{q}, t) C_{\mathbf{k}}(\mathbf{q}) + U_{ext}^*(-\mathbf{q}, t) D_{\mathbf{k}}(\mathbf{q}) \quad (21)$$

$$\langle a_{\mathbf{k}}^{t\dagger}(-\mathbf{q}) \rangle = U_{ext}^*(-\mathbf{q}, t) C_{\mathbf{k}}^*(-\mathbf{q}) + U_{ext}(\mathbf{q}, t) D_{\mathbf{k}}^*(-\mathbf{q}) \quad (22)$$

$$\langle a_{\mathbf{k}}^{t\dagger}(-\mathbf{q}) \rangle = U_{ext}^*(-\mathbf{q}, t) C_{\mathbf{k}}^*(-\mathbf{q}) + U_{ext}(\mathbf{q}, t) D_{\mathbf{k}}^*(-\mathbf{q}) \quad (23)$$

The coefficients  $C_{\mathbf{k}}(\mathbf{q})$  and  $D_{\mathbf{k}}^*(-\mathbf{q})$  satisfy,

$$\omega C_{\mathbf{k}}(\mathbf{q}) = \omega_{\mathbf{k}}(\mathbf{q}) C_{\mathbf{k}}(\mathbf{q}) + \left( \frac{v_{\mathbf{q}}}{V} \right) \Lambda_{\mathbf{k}}^{\beta}(-\mathbf{q}) \sum_{\mathbf{k}'} [\Lambda_{\mathbf{k}'}^{\beta}(-\mathbf{q}) C_{\mathbf{k}'}(\mathbf{q}) + \Lambda_{\mathbf{k}'}^{\beta}(\mathbf{q}) D_{\mathbf{k}'}^*(-\mathbf{q})] + \Lambda_{\mathbf{k}}^{\beta}(-\mathbf{q}) \quad (24)$$

$$-\omega D_{\mathbf{k}}^*(-\mathbf{q}) = \omega_{\mathbf{k}}(-\mathbf{q}) D_{\mathbf{k}}^*(-\mathbf{q}) + \left( \frac{v_{\mathbf{q}}}{V} \right) \Lambda_{\mathbf{k}}^{\beta}(\mathbf{q}) \sum_{\mathbf{k}'} [\Lambda_{\mathbf{k}'}^{\beta}(\mathbf{q}) D_{\mathbf{k}'}^*(-\mathbf{q}) + \Lambda_{\mathbf{k}'}^{\beta}(-\mathbf{q}) C_{\mathbf{k}'}(\mathbf{q})] + \Lambda_{\mathbf{k}}^{\beta}(\mathbf{q}) \quad (25)$$

Now, effective potential may be written as,

$$U_{eff}(\mathbf{q}, t) = U_{ext}(\mathbf{q}, t) + \left( \frac{v_{\mathbf{q}}}{V} \right) \langle \rho_{-\mathbf{q}} \rangle' U_{ext}(\mathbf{q}, t) \quad (26)$$

here,

$$\langle \rho_{-\mathbf{q}} \rangle = U_{ext}(\mathbf{q}, t) \langle \rho_{-\mathbf{q}} \rangle' + U_{ext}^*(-\mathbf{q}, t) \langle \rho_{-\mathbf{q}} \rangle'' \quad (27)$$

Using the fact that,

$$\langle \rho_{-\mathbf{q}} \rangle' = \sum_{\mathbf{k}} \Lambda_{\mathbf{k}}^{\beta}(-\mathbf{q}) C_{\mathbf{k}}(\mathbf{q}) + \sum_{\mathbf{k}} \Lambda_{\mathbf{k}}^{\beta}(\mathbf{q}) D_{\mathbf{k}}^*(-\mathbf{q}) \quad (28)$$

Solving these equations and using the fact that the dielectric function is just the ratio of the external divided by the effective potential we get,

$$\epsilon(\mathbf{q}, \omega) = 1 + \frac{v_{\mathbf{q}}}{V} \sum_{\mathbf{k}} \frac{n_{F,\beta}(\mathbf{k} + \mathbf{q}/2) - n_{F,\beta}(\mathbf{k} - \mathbf{q}/2)}{\omega - \frac{\mathbf{k} \cdot \mathbf{q}}{m}} \quad (29)$$

Which is, lo and behold, the RPA of Bohm and Pines.

### III. BOSE PRODUCT AND CONDENSATE-DISPLACEMENT CORRESPONDENCE

The generalisation to bose systems is entirely analogous. The answers will merely be quoted here, in fact it is easier to verify the various assertions in the bose case than it is in the fermi case. Again these formulas are valid strictly for temperatures above zero. At exactly zero temperature, the answers have been given elsewhere<sup>2</sup>.

$$b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} = n_{B,\beta}(\mathbf{k}) + \sum_{\mathbf{q}_1 \neq 0} d_{\mathbf{k}-\mathbf{q}_1/2}^{\dagger}(\mathbf{q}_1) d_{\mathbf{k}-\mathbf{q}_1/2}(\mathbf{q}_1) - \sum_{\mathbf{q}_1 \neq 0} d_{\mathbf{k}+\mathbf{q}_1/2}^{\dagger}(\mathbf{q}_1) d_{\mathbf{k}+\mathbf{q}_1/2}(\mathbf{q}_1) \quad (30)$$

and for  $\mathbf{q} \neq 0$ ,

$$b_{\mathbf{k}+\mathbf{q}/2}^{\dagger} b_{\mathbf{k}-\mathbf{q}/2} = \left( \sqrt{\frac{N}{\langle N \rangle}} \right) [\Lambda_{\mathbf{k}}^{B,\beta}(\mathbf{q}) d_{\mathbf{k}}(-\mathbf{q}) + \Lambda_{\mathbf{k}}^{B,\beta}(-\mathbf{q}) d_{\mathbf{k}}^{\dagger}(\mathbf{q})] + T_1(\mathbf{k}, \mathbf{q}) \sum_{\mathbf{q}_1 \neq 0} d_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2}^{\dagger}(\mathbf{q}_1) d_{\mathbf{k}-\mathbf{q}_1/2}(\mathbf{q}_1 - \mathbf{q}) + T_2(\mathbf{k}, \mathbf{q}) \sum_{\mathbf{q}_1 \neq 0} d_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2}^{\dagger}(\mathbf{q}_1) d_{\mathbf{k}+\mathbf{q}_1/2}(\mathbf{q}_1 - \mathbf{q}) \quad (31)$$

and the coefficients are similarly given as,

$$T_1(\mathbf{k}, \mathbf{q}) = \sqrt{(1 + n_{B,\beta}(\mathbf{k} + \mathbf{q}/2))} \sqrt{(1 + n_{B,\beta}(\mathbf{k} - \mathbf{q}/2))} \quad (32)$$

$$T_2(\mathbf{k}, \mathbf{q}) = \sqrt{n_{B,\beta}(\mathbf{k} + \mathbf{q}/2)} \sqrt{n_{B,\beta}(\mathbf{k} - \mathbf{q}/2)} \quad (33)$$

and,

$$\Lambda_{\mathbf{k}}^{\beta,B}(\mathbf{q}) = \sqrt{n_B^{\beta}(\mathbf{k} + \mathbf{q}/2)(1 + n_B^{\beta}(\mathbf{k} - \mathbf{q}/2))} \quad (34)$$

and,

$$n_B^{\beta}(\mathbf{k}) = \frac{1}{e^{\beta(\epsilon_{\mathbf{k}} - \mu_B)} - 1} \quad (35)$$

There are two points worth noting here, one is the change of sign in  $(1 + n_B^{\beta}(\mathbf{k} - \mathbf{q}/2))$  in the fermi case it was  $(1 - n_F^{\beta}(\mathbf{k} - \mathbf{q}/2))$ , the reason should be obvious. The other is the overall change in sign in  $T_2$ , this is necessary to ensure that the correct six point function is recovered. The proof that all the six-point functions come out right may be seen in Appendix.

#### IV. APPENDIX

Assume that in the fermi case the part quadratic in the sea-displacements is given by,

$$c_{\mathbf{k}+\mathbf{q}/2}^\dagger c_{\mathbf{k}-\mathbf{q}/2} |_{Quad} = \sum_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{q}_1, \mathbf{q}_2} \Gamma_{\mathbf{k}_1, \mathbf{k}_2}^{\mathbf{q}_1, \mathbf{q}_2}(\mathbf{k}, \mathbf{q}) a_{\mathbf{k}_1}^\dagger(\mathbf{q}_1) a_{\mathbf{k}_2}(\mathbf{q}_2)$$

From this we may write,

$$\begin{aligned} I &= \langle c_{\mathbf{k}+\mathbf{q}/2}^\dagger c_{\mathbf{k}-\mathbf{q}/2} c_{\mathbf{k}'+\mathbf{q}'/2}^\dagger c_{\mathbf{k}'-\mathbf{q}'/2} c_{\mathbf{k}''+\mathbf{q}''/2}^\dagger c_{\mathbf{k}''-\mathbf{q}''/2} \rangle \\ &= [(1 - n_{F,\beta}(\mathbf{k} - \mathbf{q}/2))(1 - n_{F,\beta}(\mathbf{k}' - \mathbf{q}'/2))n_{F,\beta}(\mathbf{k} + \mathbf{q}/2)\delta_{\mathbf{k}+\mathbf{q}/2, \mathbf{k}''-\mathbf{q}''/2}\delta_{\mathbf{k}-\mathbf{q}/2, \mathbf{k}'+\mathbf{q}'/2}\delta_{\mathbf{k}'-\mathbf{q}'/2, \mathbf{k}''+\mathbf{q}''/2} \\ &\quad - (1 - n_{F,\beta}(\mathbf{k} - \mathbf{q}/2))n_{F,\beta}(\mathbf{k}' + \mathbf{q}'/2)n_{F,\beta}(\mathbf{k} + \mathbf{q}/2)\delta_{\mathbf{k}-\mathbf{q}/2, \mathbf{k}''+\mathbf{q}''/2}\delta_{\mathbf{k}+\mathbf{q}/2, \mathbf{k}'-\mathbf{q}'/2}\delta_{\mathbf{k}'+\mathbf{q}'/2, \mathbf{k}''-\mathbf{q}''/2}] \quad (36) \end{aligned}$$

In terms of the Bose fields we have,

$$I = \Lambda_{\mathbf{k}}^\beta(\mathbf{q}) \Lambda_{\mathbf{k}''}^{\beta}(-\mathbf{q}'') \Gamma_{\mathbf{k}, \mathbf{k}''}^{-\mathbf{q}, \mathbf{q}''}(\mathbf{k}', \mathbf{q}') \quad (37)$$

This leads to the answers for  $T_1$  and  $T_2$  given in the main text. The arguments in the bose case is analogous.

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<sup>1</sup> G. S. Setlur, "Expressing Products of Fermi Fields in terms of Fermi-Sea displacements", UIUC preprint 1997, cond-mat/9701206, and other unpublished formulae. "Exact Momentum Distribution of a Fermi Gas in One Dimension", UIUC preprint 1997. cond-mat/9705219

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